School of Physical Sciences
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PHS 4201: ADVANCED QUANTUM MECHANICS

Assignment #3

PATH INTEGRAL FORMALISM

DATE: *27th April 2021*
DUE : *9th May 2021*

INSTRUCTIONS

- You are free to discuss the questions among yourselves if you choose to do so. However, you should write the answers independently at the end and submit them. You should be prepared to explain the steps and arguments in your answer if called upon to do so.
- Interpret the questions as a physicist and make reasonable assumptions when required and mention them.
- Your answers can be brief and to the point, giving just the essential algebraic steps and arguments. The marks for each of the questions are given on the right end of the question, in square brackets and boldface. Total marks: **45**.
- I have tried to keep the questions clear, consistent with the notation used in the class and error-free. But if you have any difficulties on these counts, feel free to email me. My email id is krthkrajeev@gmail.com .
- **While submitting the scanned copy of your assignments, please ensure that the pages of the document are legible and properly focussed.**

QUESTIONS

1. Derivation of the Simple Harmonic Oscillator (SHO) propagator. The propagator for the simple harmonic oscillator $K(x_f, t_f; x_i, t_i)$ is defined as:

$$K(x_f, t_f; x_i, t_i) \equiv \langle x_f| \exp\left[-\frac{i}{\hbar}\hat{H}(t_f - t_i)\right] |x_i\rangle\,; \qquad t_f > t_i \tag{1}$$

where, $\hat{H}$ is the Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{2}$$

(a) The Schrödinger picture:

(i) By introducing the complete set of orthonormal eigenstates $\phi_n(x)$ into Eq.(1), show that $K(x_f, t_f; x_i, t_i)$ takes the following form:

$$K(x_f, t_f; x_i, t_i) = \sum_{n=0}^{\infty} e^{-\frac{i}{\hbar}E_n(t_f - t_i)}\phi_n^*(x_f)\phi_n(x_i) \tag{3}$$

where, E_n denotes the eigenvalue corresponding to $\phi_n(x)$. **[2]**

(ii) The Hermite polynomials satisfy the following standard relation:

$$\begin{aligned}&\left(\frac{1}{\sqrt{1-\chi^2}}\right)\exp\left[\frac{-\left(\xi^2+\eta^2-2\xi\eta\chi\right)}{(1-\chi^2)}\right] \\ &= \exp\left[-\left(\xi^2+\eta^2\right)\right]\sum_{n=0}^{\infty}\left(\frac{\chi^n}{2^n n!}\right)H_n(\xi)H_n(\eta)\end{aligned} \tag{4}$$

Use the above relation to show that Eq.(3) simplifies to:

$$K(x_f, t_f; x_i, t_i) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega T)}}\exp\left[\frac{im\omega}{2\hbar\sin(\omega T)}\left\{(x_f^2 + x_i^2)\cos(\omega T) - 2x_f x_i\right\}\right] \tag{5}$$

where, $T \equiv t_f - t_i$.

[8]

(b) Path integral formalism:

(i) The path integral representation for the propagator is given by:

$$K(x_f, t_f; x_i, t_i) = \int_{x(t_i)=x_i}^{x(t_f)=x_f}\mathcal{D}[x]\exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2\right)dt\right] \tag{6}$$

A general path $x(t)$ appearing in the above integral can be written as:

$$x(t) = x_{cl}(t) + y(t); \quad y(t_i) = y(t_f) = 0, \tag{7}$$

where, $x_{cl}(t)$ is the solution to the equation of motion of SHO that satisfies the boundary condition $x_{cl}(t_i) = x_i$ and $x_{cl}(t_f) = x_f$. Show that Eq.(6) can be rewritten as:

$$K(x_f, t_f; x_i, t_i) \qquad (8)$$
$$= e^{\frac{i}{\hbar}S_{cl}(x_i,x_f,T)} \int_{y(t_i)=0}^{y(t_f)=0} \mathcal{D}[y] \exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f}\left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2\right) dt\right]$$

where, $S_{cl}(x_i, x_f, T)$ is the action evaluated for the classical trajectory $x_{cl}(t)$ and $T = (t_f - t_i)$. **[5]**

(ii) Find the explicit expression for $S_{cl}(x_i, x_f, T)$. **[10]**

(iii) Show that a general path $y(t)$ appearing in Eq.(8) can be expressed as

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\left[\frac{n\pi}{T}(t - t_i)\right]; \qquad t_i < t < t_f \qquad (9)$$

where, a_n are real constants. Assume that the path integral measure transforms as follows:

$$\int_{y(t_i)=0}^{y(t_f)=0} \mathcal{D}[y]\,(...) = Q(T)\prod_{n=1}^{\infty}\left(\frac{n\pi}{\sqrt{2}}\right)\int_{-\infty}^{\infty} da_n\,(...) \qquad (10)$$

where, $Q(T)$ is a real function and $(...)$ denotes a general functional of $y(t)$. Show that the propagator can now be written as

$$K(x_f, t_f; x_i, t_i) \qquad (11)$$
$$= e^{\frac{i}{\hbar}S_{cl}(x_i,x_f,T)} Q(T)\prod_{n=1}^{\infty}\left(\frac{n\pi}{\sqrt{2}}\right)\int_{-\infty}^{\infty} da_n \exp\left[\frac{im}{2\hbar}\frac{T}{2}\left(\frac{n^2\pi^2}{T^2} - \omega^2\right)a_n^2\right]$$

[10]

(iv) Evaluate the Gaussian integrals in Eq.(11) and use the following identity:

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{(n\pi)^2}\right) = \frac{\sin x}{x} \qquad (12)$$

to show that:

$$K(x_f, t_f; x_i, t_i) \qquad (13)$$
$$= Q(T)\left[\frac{(\omega T)}{\sin(\omega T)}\right]^{1/2} \exp\left[\frac{im\omega}{2\hbar\sin(\omega T)}\left\{(x_f^2 + x_i^2)\cos(\omega T) - 2x_f x_i\right\}\right]$$

Finally, find the explicit form of $Q(T)$ by demanding that $K(x_f, t_f; x_i, t_i)$ reduces to the free particle propagator in the limit $\omega \to 0$. **[10]**