School of Physical Sciences
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PHS 4201: Advanced Quantum Mechanics

Assignment #2

Scattering theory, Identical particles

Date: *8th April 2021*
Due : *18th April 2021*

Instructions

- You are free to discuss the questions among yourselves if you choose to do so. However, you should write the answers independently at the end and submit them. You should be prepared to explain the steps and arguments in your answer if called upon to do so.
- Interpret the questions as a physicist and make reasonable assumptions when required and mention them.
- Your answers can be brief and to the point, giving just the essential algebraic steps and arguments. The marks for each of the questions are given on the right end of the question, in square brackets and boldface. Total marks: **35**.
- I have tried to keep the questions clear, consistent with the notation used in the class and error-free. But if you have any difficulties on these counts, feel free to email me.

QUESTIONS

1 Consider the elastic scattering of a particle of mass m in a radial potential $V(r)$. Prove

$$\sigma_{\text{tot}} \approx \frac{m^2}{\pi\hbar^4} \int d^3x \int d^3x' \; V(r)\; V(r') \; \frac{\sin^2(k|\mathbf{x}-\mathbf{x}'|)}{k^2|\mathbf{x}-\mathbf{x}'|^2} \tag{1}$$

where, $\hbar k$ is the magnitude of the incident momentum and σ_{tot} is the total cross-section. You should prove the above result in two different ways:

(a) By integrating the differential cross-section computed using the first-order Born approximation. **[10]**

(b) By applying the optical theorem to the forward-scattering amplitude in the second-order Born-approximation. **[10]**

2 **Helium Atom:** Consider the Hamiltonian for a system of two electrons in the field of Helium nucleus:

$$H = \frac{\mathbf{p}_1^2}{2m_e} + \frac{\mathbf{p}_2^2}{2m_e} - \frac{1}{4\pi\epsilon_0}\frac{2e^2}{r_1} - \frac{1}{4\pi\epsilon_0}\frac{2e^2}{r_2} + \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_{12}} \tag{2}$$

where, $\mathbf{p}_i$ and $\mathbf{r}_i$ denote the momentum and position of the i-th electron and $r_{12} = |\mathbf{r}_2 - \mathbf{r}_1|$. Denote the energy eigenstate of a single electron in the helium nucleus by $\psi_{nlm}(\mathbf{r})$, where n, l, m have the usual meaning.

(a) Let us first ignore the last term in the Hamiltonian H, corresponding to the interaction between electrons. Then argue that the wave function for the system corresponding to one electron in the ground state and the other in an excited state characterised by (nlm) can be written as:

$$\phi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}}\left[\psi_{100}(\mathbf{r}_1)\psi_{nlm}(\mathbf{r}_2) \pm \psi_{nlm}(\mathbf{r}_1)\psi_{100}(\mathbf{r}_2)\right] \tag{3}$$

where, the plus sign corresponds to states with total spin quantum number $S = 1$ and minus sign corresponds to the state with total spin quantum number $S = 0$. Find the ground state of the Helium atom and the corresponding energy, when the last term in H is ignored. How does this value compare with the observed ground state energy of Helium atom, which is about $-78.8\,eV$? **[5]**

(b) Let us now consider the interaction term between the electrons as a perturbation. Find the ground state energy of Helium atom using first order time-independent perturbation theory. How does this value compare with the observed value?

You might find the following results useful:

$$\frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\gamma}} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)\;; \tag{4}$$

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^{m*}(\theta_1, \phi_1) Y_l^m(\theta_2, \phi_2) \tag{5}$$

where, $r_>(r_<)$ is the larger(smaller) of r_1 and r_2, γ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$ and $P_l(x)$ is the Legendre polynomial of degree l. **[10]**