Lecture 7: Birefringence

1 Microscopic understanding of dispersion

We have often observed that while passing through an optical medium, white light gets decomposed into its constituent wavelength. This suggests that the refractive index of the medium depends on the frequency of the electromagnetic wave passing through it. We will describe this phenomenon from a microscopic point of view, which eventually will lead us to the phenomenon of birefringence.

As an electromagnetic wave passes through an optical medium, it will exert a time dependent electric field on the atoms constituting the medium. Since magnetic fields are suppressed by a factor of c, we can ignore them for the moment being. As the external electric field is applied on an atom, it will distort the electron cloud. Again as the field disappears the electron cloud will return to its original position. We can imagine a simplified version of this set up by considering the electrons as bound to the nucleus using springs. Therefore, the consequences of the external electric field on an atom can be understood in terms of forced oscillation.

Let $E(t)$ be the sinusoidally varying external electric field with frequency ω and the characteristic frequency of vibration of the electron cloud (or, in other words, the frequency of the spring) is ω_0. Then the force equation for an electron within the electron cloud will be

$$
\begin{aligned}
& m\frac{d^2x}{dt^2} + \omega_0^2 m x = F_{\text{ext}} = qE_{\text{ext}}(t) = eE_0\cos(\omega t) \ , \\
\implies & m\frac{d^2x}{dt^2} + \omega_0^2 m x = eE_0\cos(\omega t) \ . \qquad (1)
\end{aligned}
$$

The above equation can be immediately solved by substituting, $x(t) = x_0\cos(\omega t)$, which yields,

$$
\begin{aligned}
m\left(-\omega^2\right)x_0\cos(\omega t) + \omega_0^2 m x_0\cos(\omega t) = eE_0\cos(\omega t) \ , \\
\implies x_0 = \frac{eE_0}{m(\omega_0 - \omega^2)} \ . \qquad (2)
\end{aligned}
$$

In the above equations $x(t)$ is the displacement of the electron cloud at time t due to the electromagnetic wave passing through the optical medium. As a consequence the charge center of the electron will differ from the nucleus (we will assume that the nucleus is static) and hence a dipole moment will develop. If there are N electrons, and all of them are assumed to be displaced by the same amount $x(t)$, then it follows that the total dipole moment developed will be given by

$$p = eNx = \frac{e^2 N E_0 \cos(\omega t)}{m(\omega_0^2 - \omega^2)} \equiv (\epsilon - \epsilon_0)E \;, \tag{3}$$

where the last relation defines the permittivity of the optical medium with respect to vacuum. The explicit form of the same, in terms of microscopic properties of the material, is given by,

$$\epsilon = \epsilon_0 + \frac{e^2 N}{m(\omega_0^2 - \omega^2)} \;. \tag{4}$$

The velocity of light through a medium of permittivity ϵ and permeability μ is defined as, $v = (1/\sqrt{\epsilon\mu})$. Thus the refractive index (or, its square) is given by, $n^2 = (c^2/v^2) = \epsilon/\epsilon_0$, as we assume that the material has magnetic permeability identical to the vacuum. Thus using the above relation between ϵ and frequency of the electromagnetic wave ω, the frequency dependence of the refractive index n can be expressed as,

$$n^2(\omega) = \frac{\epsilon}{\epsilon_0} = 1 + \frac{Ne^2}{m\epsilon_0} \frac{1}{(\omega_0^2 - \omega^2)} \;. \tag{5}$$

The above expression explicitly depicts that light of different wavelength will experience different refractive index and hence the phenomenon of dispersion will follow. Most interestingly, if an electromagnetic wave with frequency, $\omega \approx \omega_0$, is incident on the above medium, it will be largely absorbed and hence the flux of transmitted wave will be very small. This property will be used below to explain the phenomenon of birefringence.

2 The phenomenon of birefringence

For uniaxial crystals there is one axis which is preferred and is referred to as the optic axis. The existence of such an optic axis makes the electron clouds in an atom (disguised as oscillators) distributed in an anisotropic manner. This suggests the existence of two characteristic frequencies, say ω_{01} and ω_{02}, associated with the crystal. Suppose, ω_{01} corresponds to frequencies of the oscillators along the optic axis and ω_{02} are the frequencies of the oscillators in a plane perpendicular to the optic axis. Therefore, the refractive index along the optic axis will be different from the refractive index perpendicular to it. Hence following the previous discussion, it follows that the refractive index of a ray moving along the optic axis will be given by,

$$n_{\parallel}^2 \equiv n_{\rm o}^2 \simeq \frac{f(\omega)}{\omega_{02}^2 - \omega^2} \;, \tag{6}$$

since the polarization will be perpendicular to the optic axis and thus it will affect oscillators with frequency ω_{02}. These rays will be referred to as ordinary rays. On the other hand, for propagation in an arbitrary direction, not coincident with the optic axis, the polarization can have two parts. The component perpendicular to the plane containing optic axis and the direction of propagation will again be ordinary ray with refractive index given by the above expression. While there can also be a component of the polarization on the plane containing

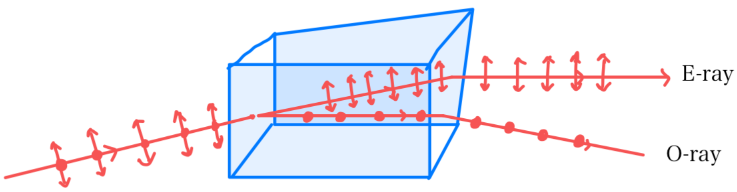

Figure 1: Formation of ordinary and extra-ordinary ray.

the optic axis and direction of polarization. The electric field arising out of this polarization will affect both types of oscillators and hence the refractive index will be,

$$n_{\mathrm{e}}^2(\omega) \simeq \frac{g_1(\omega)}{\omega_{01}^2 - \omega^2} + \frac{g_2(\omega)}{\omega_{02}^2 - \omega^2}\ , \tag{7}$$

where f, g_1 and g_2 are regular functions with no pathological behaviour at $\omega = \omega_{01}$ or, $\omega = \omega_{02}$. If an electromagnetic wave is propagating along the optic axis it will experience a single refractive index n_{o} and hence there will be no dispersion. On the other hand, for an electromagnetic wave incident on the crystal from an arbitrary direction, the electric field (or, the direction of polarization) gets decomposed into ordinary ray (with refractive index n_{o}) and extra-ordinary ray (with refractive index n within n_{o} and n_{e}). Since the refractive index $n(\omega)$ is different for these two rays, it follows that these two rays will disperse. Therefore, one will observe two sets of rays arising out of such a crystal with polarizations perpendicular to each other. This is because, the ordinary and the extraordinary rays have polarizations perpendicular to each other. For pictorial depiction, see Figure 1. In particular, if the incident electromagnetic radiation has a frequency $\omega \approx \omega_{02}$, then only the extra-ordinary ray will be transmitted and hence will be linearly polarized (the other component, namely the ordinary ray will be absorbed).

Due to the difference in the refractive index it also follows that the ordinary and the extraordinary rays will move with a different velocity. For motion of a light ray along the optic axis the velocity will correspond to that of ordinary ray (denoted by v_{o}). While for motion of the ray, such that the electric field is in the plane containing the optic axis and direction of propagation, it is denoted by v_{e}. As an example, we may consider the case of calcite crystal, one of the most prominent birefringent material. For calcite crystal it follows that, the refractive index of ordinary ray is ~ 1.7, while the refractive index of extra-ordinary ray is ~ 1.5. Therefore, $n_{\mathrm{o}} > n_{\mathrm{e}}$ for calcite crystal. Therefore, in the context of calcite crystal we further have $v_{\mathrm{o}} = (c/1.7)$ and $v_{\mathrm{e}} = (c/1.5)$ respectively. Thus in this case we have $v_{\mathrm{e}} > v_{\mathrm{o}}$ and hence we can conclude that in calcite crystal the extra-ordinary ray travels at a larger velocity than the ordinary ray (see Figure 2).

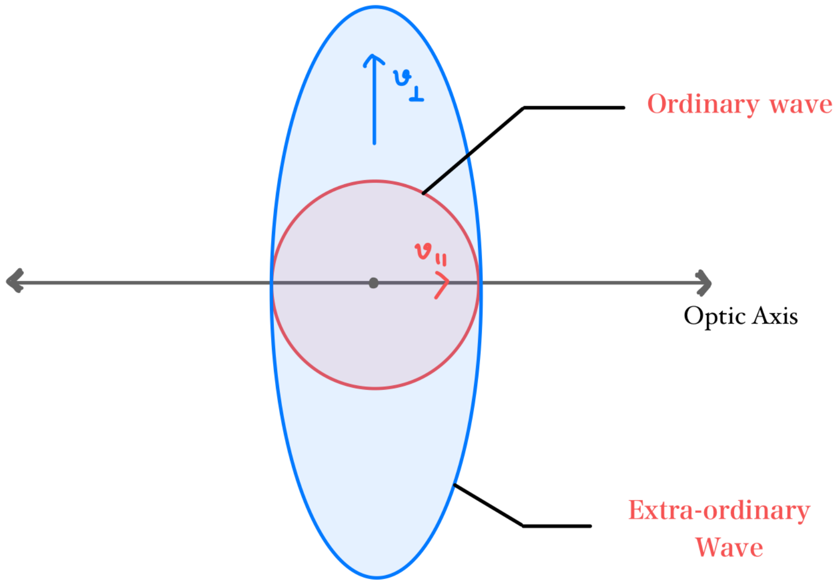

Figure 2: Different velocity for ordinary and extra-ordinary rays in a uni-axial crystal.

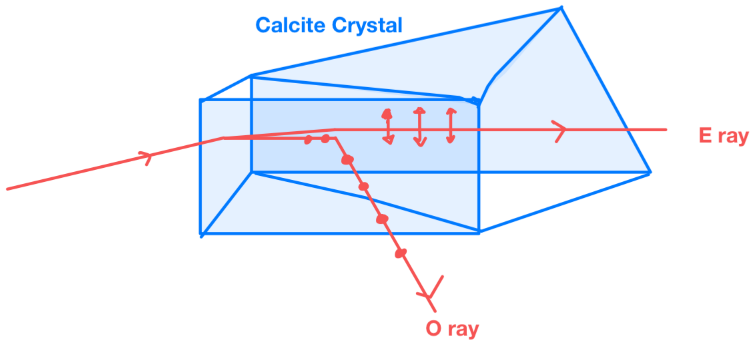

Figure 3: Total internal reflection in birefringent crystal

3 Double refraction

As emphasized in the last section, when unpolarized light is incident on crystals like calcite, quartz etc., it is refracted into two separate rays, one of which obey the laws of reflection and is called ordinary ray and the other one is referred to as the extra-ordinary ray. Since, these two rays have polarizations perpendicular to each other, using the phenomenon of birefringence one can construct linearly polarized light. This is achieved using the phenomenon of total internal reflection. In which two calcite crystals are joined together using a thin layer of adhesive material, whose refractive index is such that one of the rays, either ordinary or extra ordinary suffers total internal reflection and hence is eliminated. For calcite crystal, if one chooses the adhesive material to have a refractive index n_{ad}, such that $n_{\mathrm{o}} > n_{\mathrm{ad}} > n_{\mathrm{e}}$, then we have total internal reflection for ordinary ray while the extra-ordinary ray comes out of it. Such a scenario has been depicted in Figure 3.

To summarize, in birefringent crystals, there are two refracted rays, namely the ordinary ray and the extra-ordinary ray. The polarization of the ordinary ray is perpendicular to the plane containing the optic axis and the direction of propagation and travel with equal velocity along all the directions. While the extra-ordinary ray has polarization in the plane containing the optic axis and the direction of propagation and does not have equal velocity in all directions. However, both ordnary and extra-ordinary rays travel with the same velocity along the optic axis

In negative uniaxial crystal (e.g., calcite) the velocity parallel to the optic axis is smaller than the velocity along any perpendicular direction. On the other hand, for positive uniaxial crystal (e.g., quartz) the velocity parallel to the optic axis is larger than the velocity perpendicular to the axis. As a result, for positive uniaxial crystal we have $n_\mathrm{e} > n_\mathrm{o}$ and for the negative crystal we have $n_0 > n_e$ (see Figure 4).

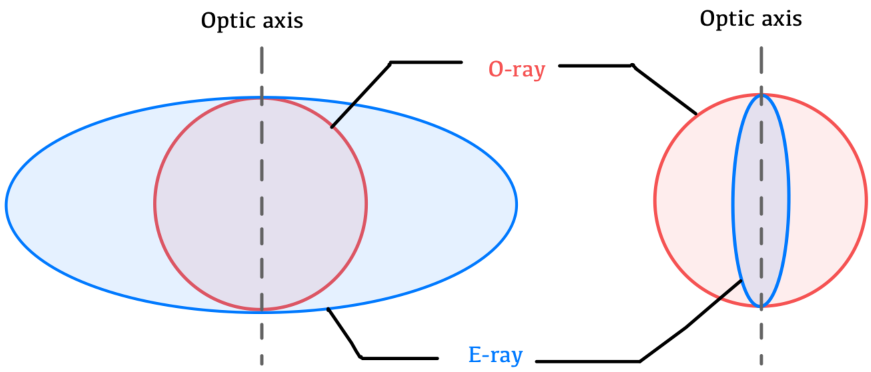

Figure 4: The left figure shows the velocity profile within a negative uniaxial birefringent crystal and the right figure shows velocity profile within a positive uniaxial crystal.

In a birefringent crystal the propagation of ordinary ray and extra-ordinary ray looks different depending upon the direction of propagation of the light ray and direction of optic axis. There are three possible cases, which can be studied — (a) the optic axis is in the plane of incidence and parallel to the refracting surface, (b) the optic axis in the plane of incidence but perpendicular to the refracting surface and (c) the optic axis is perpendicular to the plane of incidence. Waveforms of the ordinary and extra-ordinary rays in these three situations have been depicted in Figure 5. Note that for ordinary ray the velocity as well as the refractive index is same along all directions in the crystal. On the other hand, for extra-ordinary ray the velocity of the ray along the optic axis is same as that of ordinary ray, while in the direction perpendicular to the optic axis the velocity is different and is given by $v_e = c/n_e$.

This raises the natural question, as what happens to the refractive index of the extra-ordinary ray as it moves at an angle θ to the optic axis. Let the optic axis is along x direction and the direction of propagation is in the $x-y$ plane making an angle θ with the optic axis. After some time t, If propagating along the optic axis, the waveform of the extra-ordinary ray will move by

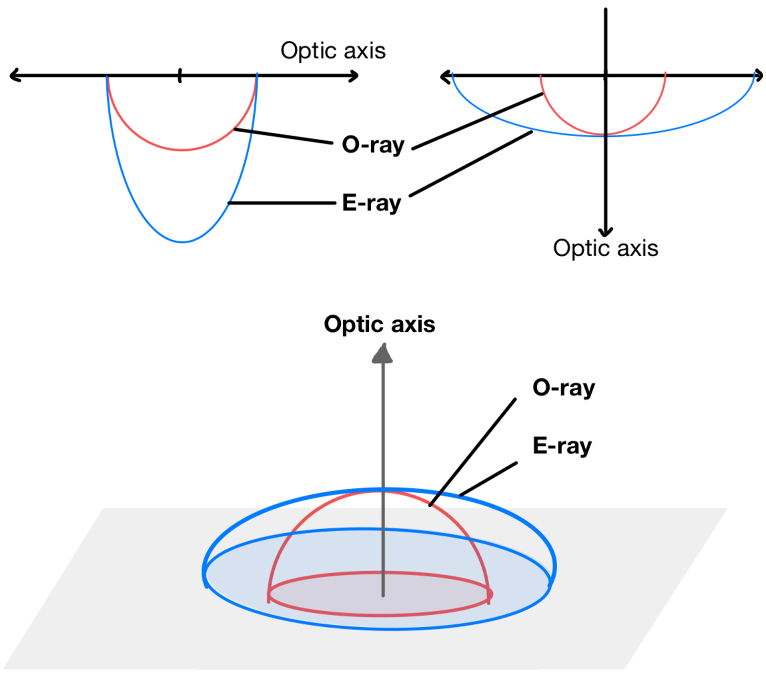

Figure 5: Waveform of ordinary ray and extra-ordinary ray in a negative uni-axial crystal for various orientations of the optic axis.

an amount $v_o t$, while in a perpendicular direction it will move by an amount $v_e t$. Thus in the $x - y$ plane the waveform will become that of an ellipse with major axis length $v_o t$ and minor axis length $v_e t$. This yields,

$$\frac{x^2}{v_o^2 t^2} + \frac{y^2}{v_e^2 t^2} = 1 \ . \tag{8}$$

We now have, $x = v_\theta \cos\theta$ and $y = v_\theta \sin\theta$, where v_θ corresponds to velocity of the extra-ordinary ray propagating in an arbitray direction θ. Thus we obtain,

$$\frac{v_\theta^2 \cos^2\theta}{v_o^2} + \frac{v_\theta^2 \sin^2\theta}{v_e^2} = 1 \ . \tag{9}$$

We further have, $\vec{v}_\theta = (v_o \cos\theta, v_e \sin\theta)$, which yields, $v_\theta^2 = v_o^2 \cos^2\theta + v_e^2 \sin^2\theta$. Substitution of this result into the above equation for ellipse finally yields,

$$\begin{aligned} \frac{1}{v_\theta^2} &= \frac{\cos^2\theta}{v_o^2} + \frac{\sin^2\theta}{v_e^2} \\ \implies \frac{c^2}{v_\theta^2} &= \frac{c^2\cos^2\theta}{v_o^2} + \frac{c^2\sin^2\theta}{v_e^2} \\ \implies n_\theta^2 &= n_o^2 \cos^2\theta + n_e^2 \sin^2\theta \ . \end{aligned} \tag{10}$$

The last relation determines the refractive index encountered by the extra-ordinary ray while propagating in a direction making an angle θ with the optic axis (see Figure 6).

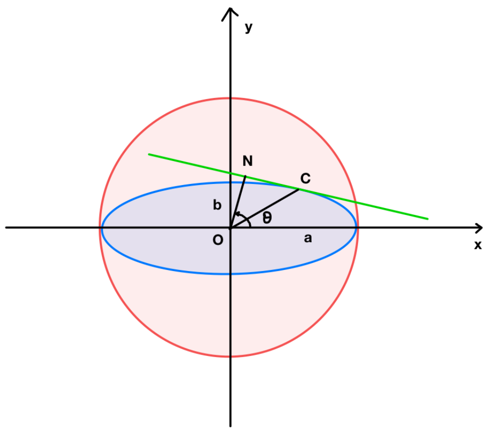

Figure 6: Propagation of extra-ordinary ray at angle θ with the optic axis.