Indian Association for the Cultivation of Science
(Deemed to be University under the *de novo* category)

Master's/Integrated Master's – PhD/PhD program

MID-Semester Examination-2019 (Autumn Semester-I/III)

Subject: Gravitation and Cosmology ***Subject Code(s): PH5102C***
Full marks: 25 ***Time allotted: 2 hrs***

Instructions

(a) Attempt any **five** questions.

(b) For most of the questions the algebra will be self-explanatory; when some interpretation/description is needed, you can keep it brief but clear. There is no need to provide extensive description.

(c) The marks for each question is given against the question. If you find that you cannot answer part (a), say, of a question but can answer part (b) *assuming the result in part (a)*, you may do so and you will get credit for part (b).

(e) You may find the following relations useful:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(-\partial_\nu g_{\alpha\beta} + \partial_\beta g_{\nu\alpha} + \partial_\alpha g_{\beta\nu} \right)$$

$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^{\beta}_{\alpha\mu} V^\mu; \qquad \nabla_\alpha V_\beta = \partial_\alpha V_\beta - \Gamma^{\rho}_{\alpha\beta} V_\rho$$

$$[\nabla_\alpha, \nabla_\beta] A^\mu = R^{\mu}{}_{\nu\alpha\beta} A^\nu; \qquad R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\beta\nu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\beta\mu}$$

1. Starting from the fact that the distance between two infinitesimally separated event is invariant under Lorentz transformation, derive the explicit transformation law between the coordinates of two inertial frames moving in the x-direction with relative velocity v. **(5 marks)**

2. Write down the coordinate transformation from Cartesian coordinates (x, y, z) to spherical polar coordinate system (r, θ, ϕ). Using these find out the metric and the non-zero components of the Christoffel symbol in the spherical polar coordinate system. **(5 marks)**

3. Write down the transformation from a Cartesian system of coordinates to that attached with a body rotating with angular velocity Ω about the z axis (assume that $v/c \ll 1$). Hence show that the distance between two infinitesimally separated events is $ds^2 = -(1 + 2\phi)dt^2 + dx^2 + dy^2 + dz^2$ and explain the physical significance of ϕ. **(5 marks)**

4. Suppose Alice is living throughout her life on the top floor of a building of height 100 meter on the surface of the earth. Her friend Bob, on the other hand, is living at the ground floor of the same building. Hence if Alice lives for 100 years, what will be the age of Bob at that instant? On the other hand, if the building is on the surface of a neutron star of mass $2M_\odot$ and radius 15 km, what will be the answer to the above question? **(5 marks)**

5. Write down the geodesic equation for a massive particle, whose trajectory is characterized by the proper time τ. Instead of the proper time τ if we use some other parameter $\lambda = \lambda(\tau)$, how the geodesic equation is modified? For what choices of $\lambda(\tau)$, it will have the same form as with τ? **(5 marks)**

6. Using the definition of covariant derivative show the following two identities:
(a) $\nabla_\alpha A^\alpha = (1/\sqrt{-g})\partial_\mu(\sqrt{-g}A^\mu)$;
(b) $\nabla_\beta A^{\alpha\beta} = (1/\sqrt{-g})\partial_\beta(\sqrt{-g}A^{\alpha\beta})$, where $A_{\alpha\beta}$ is an antisymmetric tensor.
Can you mention where the above results will become important? **(5 marks)**

7. Suppose ξ^μ is a vector generating symmetry of the spacetime. Show this vector must satisfy the condition $\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$. Can you solve this equation in flat spacetime and comment on its connection with Lorentz transformation? **(5 marks)**

8. Derive the expression for the Lie derivative of a vector field v^α along another vector field u^α, which reads $\pounds_u v^\alpha = u^\beta\partial_\beta v^\alpha - v^\beta\partial_\beta u^\alpha$. Using the fact that $\pounds_u\phi = u^\beta\nabla_\beta\phi$, determine what will be the Lie derivative of a one-form? **(5 marks)**

9. Using the definition of Riemann tensor $R_{\alpha\beta\mu\nu}$ derive the following identities — (a) $R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$; (b) $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$ and (c) $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$. **(5 marks)**