Indian Association for the Cultivation of Science
(Deemed to be university under the *de novo* category)

Master's/Integrated Master's-PhD/PhD Program

Final-Semester Examination-2019 (Autumn Semester-I/III)

Subject: Gravitation and Cosmology | **Subject Code(s): PHS 5104**
Full Marks: 50 | **Time Allotted: 3hr**

Instructions

(a) Attempt any **one** question from Part A and any **four** questions from Part B.

(b) For most of the questions the algebra will be self-explanatory; when some interpretation/description is needed, you can keep it brief but clear. There is no need to provide extensive description.

(c) The marks for each question is given against the question. If you find that you cannot answer part (a), say, of a question but can answer part (b) *assuming the result in part (a)*, you may do so and you will get credit for part (b).

Section A

attempt any one question

1. (a) Write down the transformation relation between Cartesian (x, y, z) and Cylindrical Polar (ρ, ϕ, z) coordinate system. Hence determine the components of the metric tensor in both of these coordinate systems. Using the metric tensor determine the Christoffel symbols $\Gamma^{\mu}_{\alpha\beta}$ in both these frames and compare. **(5 marks)**

   (b) An astronomical source of light is moving along a direction with velocity v making an angle θ to our line of sight. Hence show that apparent transverse speed of the source is given by,

$$v_{\text{apparent}} = \frac{v \sin\theta}{1 - (v/c)\cos\theta}$$

   where c is the speed of light. Can this velocity exceed c? If it does, will that constitute a violation of the principles of relativity? **(5 marks)**

2. (a) Suppose a particle is moving along a trajectory $x^{\mu}(\tau)$, where τ is the proper time. Then argue that the Lagrangian for the motion corresponds to $\sqrt{-g_{\mu\nu}(dx^{\mu}/d\tau)(dx^{\nu}/d\tau)}$. Hence derive the geodesic equation. **(5 marks)**

   (b) Using the geodesic equation derived above, determine how the same equation transforms as the trajectory gets parametrized by some parameter $\lambda = \lambda(\tau)$. What is the condition on the function $\lambda(\tau)$, so that the geodesic equation retains its form? **(5 marks)**

3. (a) Show that the commutator of covariant derivatives acting on a vector field is proportional to the Riemann curvature tensor. Can you comment on the scenario, when commutator of covariant derivative acts on a scalar? **(5 marks)**

   (b) Derive the Lie derivative of a scalar field and the metric tensor. Hence determine the Lie derivative of $\sqrt{-g}\phi$, where g is the determinant of the metric tensor $g_{\mu\nu}$ and ϕ is a scalar field. **(5 marks)**

Section B

Attempt any four questions

1. This question is about essential theoretical aspects of general relativity.

   (a) Demonstrate that general relativity is indeed a theory of gravity by showing that the Einstein and geodesic equations respectively reduce, in appropriate limits, to the Newtonian equation for the gravitational potential and the equation for a test mass in a weak, static gravitational field. **(4 marks)**

   (b) What property of the algebra of covariant derivatives embodied in the Ricci identity : $[\nabla_{\mu}, \nabla_{\nu}]V^{\rho} = R^{\rho}_{\sigma\mu\nu}V^{\sigma}$ leads to the uncontracted Bianchi identity

$$\nabla_{\mu}R^{\rho}_{\sigma\nu\lambda} + cyclic(\mu, \nu, \lambda) = 0 \ ?$$

   Does one need any additional relation to establish this Bianchi identity ? Derive the *contracted* Bianchi identity satisfied by the Einstein tensor from the uncontracted identity. **(6 marks)**

2. This question is about the Schwarzschild spacetime.

   (a) Starting with the Kruskal diagram for a Schwarzschild spacetime, draw the strict conformal diagram for this spacetime, clearly identifying all boundaries. For comparison, draw the strict conformal diagram for Minkowski spacetime and indicate how the singularity in Schwarzschild spacetime makes that spacetime geodesically incomplete. **(7 marks)**

(b) The black hole part of the Schwarzschild spacetime is written as $\mathcal{B} = \mathcal{M} - \mathcal{J}^{-}(\mathcal{I}^{+})$. Explain what precisely this means. What is the boundary of the region $\mathcal{B}$? **(3 marks)**

3. This question also involves the Schwarzschild spacetime.

   (a) If I am allowed to fall freely and radially from a very large distance $r_\infty >> r_s$ towards the event horizon, compute the time I will take (according to my watch) to reach the horizon. Is the time measured by me the same as measured by you who is a stationary observer at the starting point, observing my unfortunate fall ? Why ? **(5 marks)**

   (b) Inside the event horizon of a Schwarzschild black hole, what is the minimum speed for a free particle with mass, before it reaches the singularity ? Why ? **(2 marks)**

   (c) Recent observations on the star S2 orbiting the supermassive black hole Sagitarius A* in the Milky Way galaxy, have presented the star's trajectory to be a closed ellipse. How is this consistent with your understanding of bounded timelike geodesics in an exterior Schwarzschild spacetime ? **(3 marks)**

4. This question involves the *extremal* Reissner-Nordstrom black hole spacetime with the metric

$$ds^2 = \left(1 - \frac{r_0}{r}\right)^2 (dx^0)^2 - \left(1 - \frac{r_0}{r}\right)^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

   This is obtained by setting $M = Q$ in the usual RN metric.

   (a) Where is the singularity of this black hole located ? Is the singularity spacelike or timelike ? Why ? **(4 marks)**

   (b) How many regions does the horizon divide the spacetime into ? Can you identify the boundaries of these regions ? Hence, draw the conformal diagram for this black hole. Also show on this diagram a timelike geodesic starting in the asymptotic region, which traverses both the event and Cauchy horizons, avoids the singularity and reaches timelike future infinity of another asymptotic region. **(6 marks)**

5. This question is about the cosmological redshift in an FLRW spacetime.

   (a) In contrast to the derivation of the gravitational redshift in a *stationary* spacetime, discuss clearly how the isometries of the FLRW spacetime, namely spatial homogeneity and isotropy, are employed in this case, to derive the cosmological redshift formula

$$\frac{\omega_S}{\omega_O} = \frac{a(t_O)}{a(t_S)} \ ?$$

   You need to provide this derivation. **(6 marks)**

   (b) Show that for nearby galaxies, the redshift $z \equiv a(t_O)/a(t_S) - 1$ indeed obeys the linear Hubble relation with distance. **(4 marks)**

6. This question is about the Weyl tensor.

   (a) Show explicitly that the change in the affine connection $\delta_\Omega \Gamma^\rho_{\mu\nu}$ under an infinitesimal Weyl transformation $\delta_\Omega g_{\mu\nu} = \Omega g_{\mu\nu}$, transforms as a tensor under general coordinate transformations. **(2 marks)**

   (b) Hence show that $\delta_\Omega R^\rho_{\sigma\mu\nu} = \nabla_\mu(\delta_\Omega \Gamma^\rho_{\nu\sigma}) - \nabla_\nu(\delta_\Omega \Gamma^\rho_{\mu\sigma})$. **(1 mark)**

   (c) Hence derive $\delta_\Omega C^\rho_{\sigma\mu\nu}$, where $C^\rho_{\sigma\mu\nu}$ is the Weyl tensor. **(3 marks)**

   (d) Based on the conformal behaviour of the Weyl tensor and conformal diagrams, provide a brief but precise comparison (in less than a page) between the 'big bang' primordial singularity of the matter or radiation dominated FLRW spacetime, and the white hole part of the Kruskal extension of the Schwarzschild spacetime. **(4 marks)**