Indian Association for the Cultivation of Science
(a deemed to be university)

Integrated M.Sc.-Ph.D. Program in Physical Sciences (IntPPS)/PhD
(In Collaboration with Jadavpur University)

Final Examination-2019 (Spring Semester)

Paper: General Relativity and Cosmology | **Paper Code: PH-516**
Full Marks: 80 | **Time Allotted: 3hr**

Instructions

(a) Attempt any **ten** questions.

(b) For most of the questions the algebra will be self-explanatory; when some interpretation/description is needed, you can keep it brief but clear. There is no need to provide extensive description.

(c) The marks for each question is given against the question. If you find that you cannot answer part (a), say, of a question but can answer part (b) *assuming the result in part (a)*, you may do so and you will get credit for part (b).

(e) You may find the following relations useful:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(-\partial_\nu g_{\alpha\beta} + \partial_\beta g_{\nu\alpha} + \partial_\alpha g_{\beta\nu} \right)$$

$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\alpha\mu} V^\mu; \qquad \nabla_\alpha V_\beta = \partial_\alpha V_\beta - \Gamma^\rho_{\alpha\beta} V_\rho$$

$$\pounds_u v^\alpha = u^\beta \partial_\beta v^\alpha - v^\beta \partial_\beta u^\alpha; \qquad \pounds_u w_\alpha = u^\beta \partial_\beta w_\alpha + w_\beta \partial_\alpha u^\beta$$

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\alpha\rho} \Gamma^\rho_{\nu\beta} - \Gamma^\mu_{\beta\rho} \Gamma^\rho_{\nu\alpha}$$

$$R_{\alpha\beta} = \partial_\rho \Gamma^\rho_{\alpha\beta} - \partial_\alpha \partial_\beta \ln \sqrt{-g} + \partial_\rho \ln \sqrt{-g}\; \Gamma^\rho_{\alpha\beta} - \Gamma^\mu_{\alpha\rho} \Gamma^\rho_{\mu\beta}$$

1. (a) Prove that the set of all Lorentz transformations between frames moving with a relative velocity along x direction forms a group. Given the Lorentz transformation between two inertial frames moving along x-direction with relative velocity V, find out the transformation relation between velocities. **(4 marks)**

   (b) Derive the Lagrangian for a relativistic point particle by demanding the action to be invariant under Lorentz transformation. **(4 marks)**

2. (a) Derive the trajectory of a particle moving with a constant acceleration a along the x-direction. Illustrate it in a spacetime diagram. **(4 marks)**

   (b) Write down the transformation relation between Cartesian (x, y, z) and Spherical Polar (r, θ, ϕ) coordinate system. Hence determine the components of the metric tensor in both of these coordinate systems. Given the components of a vector in Cartesian coordinate system, find out the components in the Spherical polar coordinate system. **(4 marks)**

3. (a) Suppose you have a one-form (or, covariant vector) A_μ. Show that, $\partial A_\mu/\partial x^\nu$ do not transform as a tensor, but $(\partial A_\mu/\partial x^\nu - \partial A_\nu/\partial x^\mu)$ does. **(4 marks)**

   (b) Suppose Alice have kept a material, which is transparent only if light of wavelength 500 nm falls onto it, at a height of 50 meter from the ground. On the other hand, Bob is aiming a Laser light to the material from the ground. What will be the wavelength of light chosen by Bob, so that Alice can see it though the material? **(4 marks)**

4. (a)If $A^\alpha{}_{\mu\nu} B^{\mu\nu} = C^\alpha$, where C^α is a vector and $B_{\mu\nu}$ is an anti-symmetric tensor (i.e., $B_{\mu\nu} = -B_{\nu\mu}$), then show that $(A^\alpha{}_{\mu\nu} - A^\alpha{}_{\nu\mu})$ is a tensor. Can you tell anything about the symmetric part of $A_{\mu\nu}$? What happens if $B_{\mu\nu}$ is a symmetric tensor? **(4 marks)**

   (b) Suppose we have a tensor $A^{\mu\nu\alpha\beta}$ in four spacetime dimensions. This tensor is antisymmetric in the first two indices, i.e., $A^{\mu\nu\alpha\beta} = -A^{\nu\mu\alpha\beta}$ and symmetric in the last two indices, i.e., $A^{\mu\nu\alpha\beta} = A^{\mu\nu\beta\alpha}$. Determine the number of independent components this tensor has. On the other hand, If the tensor is antisymmetric in all the four indices how many independent components it will have? **(4 marks)**

5. (a) Show that $\Gamma^\alpha_{\alpha\mu} = \partial_\mu \ln\sqrt{-g}$. Hence or otherwise show that for an anti-symmetric tensor $A^{\mu\nu}$ the following result holds,

$$\nabla_\nu A^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} A^{\mu\nu}\right)$$

   What happens if $A^{\mu\nu}$ is replaced by a symmetric tensor $S^{\mu\nu}$ in the above expression? **(4 marks)**

   (b) Write down the transformation relation from Cartesian to Cylindrical polar coordinate system and hence find out how $\nabla^2\phi$ looks like in the Cylindrical polar coordinate, where ϕ is a scalar function of the coordinates. Also if $\mathbf{V}$ is a vector, what will be the expression for $\boldsymbol{\nabla} \cdot \mathbf{A}$ in Cylindrical polar coordinate? **(4 marks)**

6. (a) Suppose ξ^μ is a vector generating symmetry of the spacetime, then argue that $\pounds_\xi g_{\mu\nu}$ must vanish. Hence determine the differential equation the vector ξ^μ satisfies. If $T^{\mu\nu}$ is the conserved matter energy-momentum tensor in this spacetime, show that $\nabla_\mu (T^{\mu\nu}\xi_\nu) = 0$. What does it signify? **(4 marks)**

   (b) Find out the Lie derivative of a vector and a scalar from first principles. Hence determine the Lie derivative of a one-form. **(4 marks)**

7. (a) Show that, using coordinate transformation, spacetime metric $g_{\mu\nu}$ at a point can always be reduced to $\eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu}$ can be set to zero. But all second derivatives of the metric cannot be set to zero and the number of remaining components will be 20 in four dimensions. **(4 marks)**

(b) Derive geodesic equation with affine parameter τ, starting from the action,

$$\int d\tau \left\{ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}$$

(4 marks)

8. (a) Derive the expression for curvature tensor in terms of Christoffel symbols by parallel transporting a vector field along a closed curve. **(4 marks)**

   (b) Show that the curvature tensor $R_{\mu\nu\alpha\beta}$ has the following properties — (a) $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$, (b) $R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0$ and (c) $\nabla_\sigma R_{\mu\nu\alpha\beta} + \nabla_\beta R_{\mu\nu\sigma\alpha} + \nabla_\alpha R_{\mu\nu\beta\sigma} = 0$. **(4 marks)**

9. (a) Starting from the gravitational action vary the metric to derive Einstein's field equations. Comment on the boundary value problem. **(4 marks)**

   (b) Given the electromagnetic Lagrangian $-(1/16\pi)F_{\mu\nu}F^{\mu\nu}$, find out the energy-momentum tensor associated with it. **(4 marks)**

10. (a) Prove that the energy-momentum tensor derived from varying the matter Lagrangian is conserved, provided the field equations for the matter field hold true. **(4 marks)**

    (b) Starting from the Lagrangian of a scalar field with a potential derive the associated energy-momentum tensor. Show that it is symmetric and hence demonstrate that it is conserved provided the field equation for the scalar field holds true. **(4 marks)**

11. (a) Given the Schwarzschild solution, find out the conserved quantities associated with the geodesic motion. Using these write down the geodesic equation on the equatorial plane. **(4 marks)**

    (b) Assuming that the orbit is nearly circular, show that the orbit precesses and find out the precession angle. **(4 marks)**

12. (a) Starting from the Schwarzschild spacetime, define the tortoise coordinate. Hence introduce the advanced and retarded null coordinates and write down the form of the Schwarzschild metric in those coordinates. Finally, define the Kruskal coordinates and express the Schwarzschild spacetime in Kruskal coordinates. Is there any singularity at $r = 2M$? **(4 marks)**

    (b) Draw the Penrose diagram of Minkowski spacetime and identify particles moving with uniform velocity with respect to one another. **(4 marks)**

13. (a) Find out the geodesic equation of a massless particle in the Schwarzschild spacetime. Hence determine the angle by which a light ray will bend as it passes close to a massive object. **(4 marks)**

    (b) Draw the Penrose diagram of a Schwarzschild spacetime and identify particles falling into the black hole as well as remaining stationary at a fixed radius. **(4 marks)**

14. (a) Argue that isotropy and homogeneity of our universe demands the spacetime metric to be expressed as $g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t))$. As a consequence determine how the wavelength of a photon changes? **(4 marks)**

    (b) Given the energy momentum tensor of a perfect fluid as, $T^\mu_\nu = \text{diag}(-\rho, p, p, p)$, find out the conservation equation derived from $\nabla_\mu T^\mu_\nu = 0$ in a homogeneous and isotropic background. Using one of the Einstein's equation $H^2 = (8\pi G/3)\rho$ and the above conservation equation derive the other Einstein's equation, $2(\ddot{a}/a) + (\dot{a}^2/a^2) = -8\pi G p$. **(4 marks)**

15. (a) Given a perfect fluid with a constant equation of state $\omega = p/\rho$, find out how the energy density varies with time. Use the appropriate choices of ω for matter and radiation dominated universe and hence find the time evolution. **(4 marks)**

    (b) Find out the time dependence of scale factor when both matter and radiation is present, but cosmological constant can be ignored. Comment on the limits when a is large and when it is small. **(4 marks)**