SPS and SMCS
IACS
Kolkata 700032, India

Integrated Master's-PhD Programme
Second Year, Fourth Semester
2019-2020

PHS – 5207: PHYSICS OF BLACK HOLES & GRAVITATIONAL WAVES

Assignment #1 - Congruences and Hypersurfaces

UNDERSTANDING PROPERTIES OF CONGRUENCES AND HYPERSURFACES

Sumanta Chakraborty

DATE: *15th January 2020*
DUE : *24th January 2020*

INSTRUCTIONS

- You are free to discuss the questions among yourselves if you choose to do so. However, you should be able to explain all the steps and arguments in order to arrive at the desired answer if called upon to do so.
- Interpret the questions as a physicist; not as a mathematician. Make reasonable assumptions when required and mention them.
- Your answers can be brief and to the point, giving just the essential algebraic steps and arguments. The marks for each of the questions are given at the right end of the question. Total marks: **65**.
- I have tried to keep the questions clear, consistent with the notation used in the class and error-free. But if you have any difficulties on these counts, feel free to drop me an email.

1. **Raychaudhuri Equations for timelike geodesics** — We often talk about Raychaudhuri equation, which depicts how the expansion θ changes along the geodesic. This has also been derived in the class starting from the trace of the expression $u^c\nabla_c B_{ab} = -R_{ambn}u^m u^n - B_{ac}B^c{}_b$. But there are actually two additional equations which are also referred to as the Raychaudhuri equations. You will derive these equations, in this exercise,

   - Consider the traceless and symmetric part of the above equation for $u^c\nabla_c B_{ab}$ and hence determine the following evolution equation for the shear tensor σ_{ab}, such that,

   $$u^c\nabla_c\sigma_{ab} = -\frac{2}{3}\theta\sigma_{ab} - \sigma_{ac}\sigma^c{}_b - \omega_{ac}\omega^c{}_b + \frac{1}{3}\left(\sigma^{pq}\sigma_{pq} - \omega^{pq}\omega_{pq}\right)h_{ab}$$
   $$- W_{ambn}u^m u^n + \frac{1}{2}R^{\mathrm{TT}}_{ab}\,,$$

     where R^{TT}_{ab} is the transverse traceless part of the Ricci tensor. This is defined as, $R^{\mathrm{TT}}_{ab} = R^{\mathrm{T}}_{ab} - (1/3)R^{\mathrm{T}}h_{ab}$, where $R^{\mathrm{T}}_{ab} = h^c_a h^d_b R_{cd}$ and $R^{\mathrm{T}} = h^{pq}R^{\mathrm{T}}_{pq}$ is the trace of the projected Ricci tensor. Further, W_{abcd} is the Weyl tensor. **[6]**

   - Compute the antisymmetric part of the equation for $u^c\nabla_c B_{ab}$ and hence determine the following evolution equation for the rotation tensor ω_{ab} as,

   $$u^c\nabla_c\omega_{ab} = -\frac{2}{3}\theta\omega_{ab} - \sigma_{ac}\omega^c{}_b - \omega_{ac}\sigma^c{}_b\,.$$

     Thus in total, $(d\theta/d\tau)$, $u^c\nabla_c\sigma_{ab}$ and $u^c\nabla_c\omega_{ab}$ constitute the Raychaudhuri equations. **[4]**

2. **Practice with timelike congruence** — Consider the following vector field in the Schwarzschild spacetime,

   $$u^a = \left(\delta^a_t + \sqrt{\frac{M}{r^3}}\delta^a_\theta\right)$$

   where (t, r, θ, ϕ) are the usual Schwarzschild coordinates and M is the mass of the black hole. Normalize the vector field and use the normalized vector field in order to work out the following problems:

   - Show that the vector field is time-like and geodesic. Describe the geodesic curves to which u^a is tangent. **[3]**

   - Calculate the expansion of the congruence with the normalization. Explain why the expansion is positive in the northern hemisphere, while negative in the southern hemisphere. Explain also why the expansion is singular at both north and south poles. **[5]**

- Compute the rotation tensor for this congruence and show that the square of the rotation tensor has the form,

$$\omega_{ab}\omega^{ab} = \frac{M}{8r^3}\left(\frac{1-\frac{6M}{r}}{1-\frac{3M}{r}}\right)^2$$

[4]

- Calculate $(d\theta/d\tau)$ as well as shear tensor σ_{ab} and check that Raychaudhuri's equation is satisfied. **[3]**

3. **Practice with null congruence** — Consider a spacetime with the following metric,

$$ds^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + \left(r^2 + a^2\cos^2\theta\right)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2$$

where a is a constant and consider the following vector field,

$$\ell^a = \delta^a_t + \delta^a_r + \frac{a}{r^2 + a^2}\delta^a_\phi$$

Hence verify the following results:

- Check that ℓ^a is null and it satisfies the geodesic equation. Also depict that r is an affine parameter of this null geodesic. **[3]**
- Find an auxiliary null vector k^a and calculate the expansion, shear and rotation of the null congruence whose tangent is given by k^a. Hence verify the following results:

$$\theta = \frac{2r}{r^2 + a^2\cos^2\theta}; \qquad \sigma_{ab} = 0; \qquad \omega_{ab}\omega^{ab} = \frac{2a^2\cos^2\theta}{(r^2 + a^2\cos^2\theta)^2}$$

Thus argue that the congruence is diverging, shear free and not hypersurface orthogonal. **[5]**

- Show that the coordinate transformation,

$$x = \sqrt{r^2 + a^2}\,\sin\theta\cos\phi; \qquad y = \sqrt{r^2 + a^2}\,\sin\theta\sin\phi; \qquad z = r\cos\theta$$

brings the metric to the standard Minkowski form for flat spacetime. Express the null vector ℓ^a in this coordinate system. **[2]**

4. **Raychaudhuri equation with non-affine parametrization** — We have used affinely parametrized null geodesics in order to derive the Raychaudhuri equation for null congruence. In this exercise, you will derive the Raychaudhuri equation for non-affinely parametrized null geodesics. If ℓ^a is non-affinely parametrized, then $\ell^a\nabla_a\ell^b = \kappa\ell^b$ and κ is called the non-affinity parameter.

- Show that even with non-affine parametrization we have,
$$q^a_c q^b_d \left(\nabla_a \ell_b\right) = \nabla_c \ell_d + \ell_c k^p \left(\nabla_p \ell_d\right) + \ell_d \left(k^p \nabla_c \ell_p\right) + \ell_c \ell_d \left(k^p k^q \nabla_p \ell_q\right)$$
[3]
- Show that the expansion of the null congruence is given by, $\theta = \nabla_a \ell^a - \kappa$. Find out how σ_{ab} and ω_{ab} changes as non-affine parametrization is considered. **[5]**
- Show that the Raychaudhuri equation now takes the following form,
$$\frac{d\theta}{d\lambda} = \kappa\theta - \frac{1}{2}\theta^2 - \sigma^{ab}\sigma_{ab} + \omega^{ab}\omega_{ab} - R_{ab}\ell^a\ell^b$$
[2]

5. **First step towards black hole thermodynamics** — In this exercise you will take the first step towards understanding black hole thermodynamics. Consider Schwarzschild spacetime and r = constant surface in the spacetime. Hence work out the following problems:
   - Determine the normal to the r = constant surface and normalize it. Then find out the expansion of the normalized normal, n^a, which is defined as $K = \nabla_a n^a$. This is the trace of the extrinsic curvature associated with the surface. Determine what happens to K in the limit $r \to 2M$. **[3]**
   - Also find out the induced metric on the r = constant surface and hence determine its determinant. What happens to the determinant in the limit $r \to 2M$. Thus find out the combination $2K\sqrt{h}$ and hence determine its limit as $r \to 2M$. **[3]**
   - Divide the final expression for $2K\sqrt{h}$ on the $r = 2M$ surface by $16\pi G$ and provide physical implication of the same. **[1]**
   - Execute the above steps for a general static and spherically symmetric metric, $ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2$, where $f(r)$ vanishes at a certain radius $r = r_{\mathrm{h}}$. Hence determine $2K\sqrt{h}$ over the surface $r = r_{\mathrm{h}}$ and provide physical implications. **[3]**

6. **Freely falling observer in Schwarzschild** — In this exercise, we will work out the properties of a specific space-like hypersurface in Schwarzschild spacetime. Consider the T = constant hypersurface in Schwarzschild spacetime, where
$$T = t + 4M\left[\sqrt{\frac{r}{2M}} + \frac{1}{2}\ln\left(\frac{\sqrt{r/2M} - 1}{\sqrt{r/2M} + 1}\right)\right] ,$$
where t and r are the standard Schwarzschild coordinates. Further, (r, θ, ϕ) are used as coordinates intrinsic to the hypersurface. Hence work out the following exercises,

- Calculate the unit normal n_a and the induced metric h_{ab} in the Schwarzschild coordinates. **[3]**
- Compute $h_a^c \nabla_c n_b$, which is called the extrinsic curvature of the hypersurface. Hence determine the trace. Is the extrinsic curvature and/or its trace is finite at $r = 2M$. **[4]**
- Express the Schwarzschild metric in terms of the coordinates (T, r, θ, ϕ) and show that the metric is regular at $r = 2M$. **[3]**