SPS and SMCS
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**PHS – 2201: CLASSICAL MECHANICS & SPECIAL RELATIVITY**

# Assignment #1 - Lagrangian Formulation in Classical Mechanics

UNDERSTANDING LAGRANGIAN MECHANICS AND RELATED MANIPULATIONS

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## INSTRUCTIONS

- You are free to discuss the questions among yourselves if you choose to do so. However, you should be able to explain all the steps and arguments in order to arrive at the desired answer if called upon to do so.
- Interpret the questions as a physicist; not as a mathematician. Make reasonable assumptions when required and mention them.
- Your answers can be brief and to the point, giving just the essential algebraic steps and arguments. The marks for each of the questions are given at the right end of the question. Total marks: **60**.
- I have tried to keep the questions clear, consistent with the notation used in the class and error-free. But if you have any difficulties on these counts, feel free to drop me an email.

1. **Particle on a Sphere** — Consider a particle of mass $m$ moving under the force of gravity on the surface of a smooth sphere of radius $R$. Find out the Lagrangian and the equation of motion. Hence find out the angle at which the particle will leave the surface of the sphere. **[5]**

2. **Schrödinger equation from Action principle** — The Schrödinger equation can also be derived using Action principle. For that consider the following action,

$$\mathcal{A} = \int d^3\boldsymbol{x}\ \psi^*(\boldsymbol{x})H(\boldsymbol{x},\boldsymbol{p})\psi(\boldsymbol{x})$$

where $H(\boldsymbol{x},\boldsymbol{p})$ is the Hamiltonian operator of the system. We assume that following form for the Hamiltonian operator, $H(\boldsymbol{x},\boldsymbol{p}) = -(\hbar^2/2m)\boldsymbol{\nabla}^2 + V(\boldsymbol{x})$. Show that the variation of the above action $\mathcal{A}$ with respect to $\psi(\boldsymbol{x})$ and $\psi^*(\boldsymbol{x})$ yields, the Schrödinger equation, provided we impose the following constraint, $\psi(\boldsymbol{x})$ is normalized. **[5]**

3. **Simple Pendulum** — In this problem we will discuss various aspects of a simple pendulum of mass $m$ using Lagrangian formulation. In particular, when the point of suspension of a simple pendulum is executing certain motion.

   - First of all write down the Lagrangian of a simple pendulum with mass $m$ and find out the equation of motion. **[5]**
   - Suppose, the point of suspension of the harmonic oscillator has a mass $\bar{m}$ and is moving along horizontal direction. Find out the Lagrangian and associated equation of motion. **[5]**
   - The point of suspension of mass $\bar{m}$ is moving (a) on a vertical circle with constant frequency $\omega$, (b) along a horizontal line with constant frequency $\omega$ and (c) along a vertical line with constant frequency $\omega$. Find out the Lagrangian and the equation of motion in each of these cases. **[10]**

4. **Damped Harmonic Oscillator** — If the Lagrangian does not explicitly depends on time, then the energy is conserved. This is the case for Harmonic oscillator. Write down the Lagrangian for harmonic oscillator and explicitly verify the same. However, the energy is certainly not conserved for a damped harmonic oscillator. Thus the Lagrangian must depend explicitly on time. This gives us the following choice for the Lagrangian $L = f(t)(m/2)\dot{q}^2 - g(t)(m/2)\omega^2 q^2 + h(t)$. Can you determine the unknown functions $f(t)$, $g(t)$ and $h(t)$ for which the equation of motion for the damped harmonic oscillator can be obtained? Does the presence of $h(t)$ affects the field equations? Explain. **[10]**

5. **Dissipative Forces and Electric Circuit** — The form of the Lagrangian $L = T - V$ is for conservative forces. Sometimes dissipative forces, e.g., damped harmonic oscillator can be accounted for by introducing time dependent functions in the Lagrangian. However, such is not the case always. The dissipative forces are generally accounted for, by modifying the Lagrange's equations to,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right) + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

where $\mathcal{F}$ is called Rayleigh's dissipation function, which is a function of velocities and possibly of coordinates. Determine the form of $\mathcal{F}$ for standard dissipative forces, which are proportional to the velocity. **[3]**

This set of Lagrange's equations can be applied to electrical circuits in order to determine how the electric current is changing with time. Suppose, you have an electric circuit with resistance $R$ and inductance $L$ in series with a voltage difference across the circuit being $V$. Determine the Lagrangian and the associated equation of motion. Solving the same find out the time variation of the electric current. On the other hand, if we have inductance $L$ and capacitance $C$ in series in an electrical circuit with voltage difference $V_0 \cos\omega t$, find out the time variation of the electric charge. For which value of $\omega$, the system can be said to be in resonance? **[7]**

6. **Lagrangian with higher derivatives** — A very pertinent question in the Lagrangian formulation of any system is "Why does the Lagrangian does not contain more than one temporal derivative of the dynamical variable?". In this exercise you will solve for the first part of the above question, which is very intimately connected with a famous instability in any theory involving higher derivatives, known as the Ostrogradsky instability. To understand this consider a Lagrangian $L = L(q, \dot{q}, \ddot{q})$. Hence work out the followings:

   - Vary the action, which is the integral of the above Lagrangian and obtain the Euler-Lagrange equation of motion for $q(t)$. What are the boundary conditions? Are they consistent with the equation of motion? **[5]**
   - Take the following Lagrangian, $L = -(m/2)q\ddot{q} - (k/2)q^2$, where $m$, $k$ are constants and determine the equation of motion. Is the equation of motion, consistent with boundary data? **[2]**
   - Does the equation of motion appear familiar, i.e., can you derive the same equation of motion starting from another Lagrangian? How does the Lagrangian you have constructed differs from the above Lagrangian? Can you relate to the boundary data you have to specify at the end points? **[3]**